A CHARACTERIZATION AT INFINITY OF BOUNDED VORTICITY, BOUNDED VELOCITY SOLUTIONS TO THE 2D EULER EQUATIONS

JAMES P. KELLIHER

ABSTRACT. We characterize the possible behaviors at infinity of weak solutions to the 2D Euler equations in the full plane having bounded velocity and bounded vorticity. We show that any such solution can be put in the form obtained by Ph. Serfati in 1995 after a suitable change of reference frame. Our results build on those of a recent paper of the author's, joint with Ambrose, Lopes Filho, and Nussenzveig Lopes.

Includes some proofs and additional details not intended for submission. These details appear in blue in small font.

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1. Introduction

In classical form, the Euler equations (without forcing) can be expressed as

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p &= 0, \\ \operatorname{div} u &= 0, \\ u(0) &= u^0. \end{cases} \tag{1.1}$$

Here, u is a velocity field, p is a scalar pressure field, and the initial velocity, u^0, is assumed to be divergence-free. We are concerned here exclusively with solutions in the full plane.

The nature of the solutions to these equations will depend strongly on the function spaces to which the initial data belongs. For functions spaces for which well-posedness results are known, nearly all studies have assumed that the vorticity, $\omega = \operatorname{curl} u := \partial_1 u^2 - \partial_2 u^1$, decays at infinity rapidly enough that the velocity can be recovered from the vorticity via the Biot-Savart law,

$$u = K * \omega,$$

where K is the Biot-Savart kernel (see (2.1)). One commonly imposed condition that insures this is that $\omega \in L^{p_1} \cap L^{p_2}$ for some $p_1 < 2 < p_2$, in which case the velocity will also decay at infinity. (The Biot-Savart law can hold with some decay of the vorticity but without decay of the velocity at infinity, and solutions to the Euler equations can still be obtained: see [2].)

We will be concerned here with initial data for which the Biot-Savart law does not hold, treating the case where the vorticity and velocity are both bounded: what we call *bounded* solutions. The construction of such solutions in the full plane was first decribed by Ph. Serfati in [16], proven in more detail in [1] (including the case of an exterior domain). An alternate construction, relying upon another Serfati paper, [17], was given by Taniuchi in [18].

In each of [16, 18, 1], however, the behavior at infinity of a solution was assumed either implicitly or explicitly. Identical assumptions, on the velocity, are made in [16, 1], while [18] makes an assumption on the pressure. (We describe these assumptions in detail below.) These assumptions are a priori, in that they are used in the construction of the solutions. The purpose of this work is to characterize a postierori all possible behaviors of bounded solutions at infinity, so as to avoid the need for such assumptions a priori.

To understand what types of behavior at infinity we might expect, consider the following two classical solutions (u_1, p_1) and (u_2, p_2) to (1.1):

$$\begin{aligned} u_1(t,x) &= u^0 + U_\infty(t), & p_1(t,x) &= -U_\infty'(t) \cdot x, \\ u_2(t,x) &= u^0, & p_2(t,x) &= 0. \end{aligned} \tag{1.2}$$

Here, U_∞ is any differentiable vector-valued function of time for which $U_\infty(0) = 0$. Both are easily verified to be solutions to the Euler (and, for that matter, Navier-Stokes) equations as in (1.1) with the same initial velocity, u^0. In [11, 13], the authors use these examples to make the point that to insure solutions are unique, some condition on the pressure must be imposed for solutions to the Navier-Stokes equations in the plane.

Here, we draw a different lesson from this example, one that leads to a characterization of all possible bounded solutions to the Euler equations. We prove that any solution's behavior at infinity is of necessity very much like that of (u_1, p_1).

Specifically, for solutions in the full plane, we show that there exists some continuous vector-valued function of time, U_∞, with $U_\infty(0) = 0$, for which

$$\begin{aligned} u(t,x) - u(0,x) &= U_\infty(t) + \lim_{R\to\infty} (a_R K) * (\omega(t) - \omega(0))(x), \\ \nabla p(t,x) &= -U_\infty'(t) + O(1), \\ p(t,x) &= -U_\infty'(t) \cdot x + O(\log|x|), \end{aligned} \tag{1.3}$$

the explicit expression for the $O(1)$ (in $|x|$) function being given in (2.5). In (1.3), $\omega(t) = \partial_1 u^2(t) - \partial_2 u^1(t)$ is the vorticity (scalar curl) of the velocity field $u(t)$, K is the Biot-Savart kernel (see (2.1)), and a_R is any cutoff function with support increasing to infinity with R, as in Definition 2.6. The time derivative on U_∞ in $(1.3)_{2,3}$ is a distributional derivative.

To explain what $(1.3)_1$ means, we need one basic fact concerning the Biot-Savart law: If $\omega \in L^1 \cap L^\infty(\mathbb{R}^2)$ then $u = K * \omega$ is the unique, divergence-free vector field vanishing at infinity whose vorticity is ω.

The condition that ω be in $L^1 \cap L^\infty$ can be weakened, but some decay at infinity is required for the Biot-Savart law to hold. Hence, we have no hope of applying the Biot-Savart law for our solutions, as we wish to assume no decay of vorticity. But we will discover a replacement for the Biot-Savart law that will work, and name it the *renormalized Biot-Savart law*, defined as follows:

> We say that the renormalized Biot-Savart law holds for a vector field, v, if there exists a constant vector field, H, such that
>
> $$v = H + \lim_{R\to\infty} (a_R K) * \omega(v) \tag{1.4}$$
>
> pointwise in $\mathbb{R}^2$, where $\omega(v) := \partial_1 v^2 - \partial_2 v^1$.

When $\omega(v)$ has sufficient decay at infinity, (1.4) holds without the need for a cutoff function: we simply obtain $v = H + K * \omega$, with H being the value of v at infinity.

The relation in $(1.3)_1$, then, says that the renormalized Biot-Savart law holds for the vector field $u(t) - u(0)$ at any time, t, with $H = U_\infty(t)$.

The velocity field, U_∞, can be eliminated in (1.3) (or in $(1.2)_1$) by changing to an accelerated frame of reference by the transformation,

$$\begin{aligned} \overline{x} = \overline{x}(t,x) &= x + \int_0^t U_\infty(s)\, ds, \\ \overline{u}(t,x) = u(t,\overline{x}) - U_\infty(t), &\quad \overline{p}(t,x) = p(t,\overline{x}) + U_\infty'(t) \cdot x. \end{aligned} \tag{1.5}$$

(See the first part of Lemma 6.1.) Note that this is a Galilean transformation when U_∞ is constant in time. Setting $\overline{\omega} = \omega(\overline{u})$, the chain rule gives $\overline{\omega}(t, x) = \omega(t, \overline{x})$, and it follows that

$$\begin{aligned} \overline{u}(t, x) - \overline{u}(0, x) &= \lim_{R \to \infty} (a_R K) * (\overline{\omega}(t) - \overline{\omega}(0))(x), \\ \nabla \overline{p}(t, x) = O(1), &\quad \overline{p}(t, x) = O(\log |x|), \end{aligned} \tag{1.6}$$

and $(\overline{u}, \overline{p})$ satisfy the Euler equations in the sense of distributions. Physically, this reflects the fact that a change of frame by translation, even an accelerated translation, introduces a force that is a gradient, and so is absorbable into the pressure gradient.

Alternately, we can view solutions for which U_∞ is not identically zero to be in an accelerated frame: we then move to an inertial frame, in which $U_\infty \equiv 0$, by the transformation above. Such solutions in an inertial frame are identical to those constructed by Serfati in [16]. Observe as well that the two solutions in (1.2) are the same solution after the transformation in (1.5).

That U_∞ can be eliminated by changing frames in this way is an a posteriori conclusion reached only after establishing the existence of such a vector field for which (1.3) holds. Since we cannot transform U_∞ away until we obtain it, obtaining it is unavoidable. Moreover, it is in demonstrating that (1.3) must hold for some U_∞ that we say we *characterize* solutions to the Euler equations at infinity.

To cast a different light on our characterization of solutions, consider the special case of sufficiently decaying (say, compactly supported) initial vorticity in the full plane. Then the classical Biot-Savart law applies, and $(1.3)_1$ reduces to $u(t) = U_\infty(t) + K * \omega(t)$. This gives the usual characterization of solutions to the 2D Euler equations for decaying vorticity whose velocity at infinity is U_∞ (often chosen to be zero). Actually, this is not normally viewed as a characterization of the solution, but rather as a way of recovering the velocity from the vorticity, and so obtaining a formulation of the Euler equations solely in terms of the vorticity. This same point of view applies for our non-decaying bounded solutions as well.

Key to our characterization of the velocity field for a solution, u, to the 2D Euler equations in the full plane is the observation that any bounded velocity field, v, having bounded vorticity satisfies the renormalized Biot-Savart law (1.4) for a subsequence (see Lemma 2.8). Applying this to $v = u(t) - u(0)$ and using properties of the Euler equations allows us to show that $(1.3)_1$ holds.

Having obtained the characterizations in $(1.3)_1$, the task of establishing existence and uniqueness immediately arises. We will find this task easy, however, because existence and uniqueness in the special case of $U_\infty \equiv 0$ was already proved in [1] (for both the full plane and the exterior of a single obstacle). The transformation in (1.5) makes this especially simple.

The characterizations in $(1.3)_1$ along with existence and uniqueness give a fairly complete picture of the velocity for bounded solutions to the Euler equations. For the pressure, we take a much different approach, for we will not find it possible to directly characterize the pressure as we did the velocity. Limiting us in this regard is the lack of decay at infinity of the velocity field (from which the pressure is ultimately derived).

Instead, we will show that the solutions we construct in our proof of existence also satisfy $(1.3)_{2,3}$. We do this using the sequence of smooth approximate solutions, which decay sufficiently rapidly at infinity, and taking a limit. Because we have uniqueness of solutions using only $(1.3)_1$, it follows that $(1.3)_{2,3}$ hold for all bounded solutions. (See [13] for another approach to dealing with the pressure in the setting of the Navier-Stokes equations for bounded velocity.)

Let us call a divergence-free bounded velocity field having bounded vorticity a *Serfati velocity*. The question of whether a given bounded vorticity has an associated Serfati velocity is a delicate one. A number of examples are given in [1]: these include some obvious examples, such as doubly-periodic vorticity integrating to zero on its fundamental domain and vorticity in $L^1 \cap L^\infty$, as well as some less obvious ones, such as the characteristic function of an infinite strip. Asking whether a bounded initial velocity is Serfati is a less delicate question, as one need only compute its scalar curl. Any Lipschitz divergence-free vector field is a Serfati velocity. From such an initial velocity one can obtain a unique solution, but with only the vorticity bounded. The same is true for initial velocity in C^1, but existence and uniqueness in $C_{loc}(\mathbb{R}; C^{1,\alpha})$ was shown in[17].

We say now a few words about works in the literature pertaining to bounded solutions to the 2D Euler equations and how they relate to this work.

Our proof of the existence and uniqueness of solutions in Section 6 is a modest extension of the proof in [1], which in turn builds on the approach in [16], where the existence and uniqueness of such solutions was first proved by Serfati in the full plane. Serfati's full-plane existence result was extended by Taniuchi in [18] to allow slightly unbounded vorticity (a localized version of the velocity fields treated by Yudovich in [22]), while Taniuchi with Tashiro and Yoneda in [19] established uniqueness (and more). In [1], Serfati's result was obtained both for the full plane and for the exterior to a single obstacle.

In each of these papers, the solutions that are constructed have a special property that is used as a selection criterion to guarantee uniqueness. In [18, 19], that property is that the pressure belong to BMO and is given by a Riesz transform in the classical way. (This implies at most logarithmic growth of the pressure at infinity, as we show.) In [1], an identity ((2.2), below, with $U_\infty \equiv 0$) that we show is equivalent to $(1.3)_1$ is used. This identity, called the *Serfati identity* here and in [1], is implicitly used, though never explicitly stated, by Serfati both in the construction of a solution (in the full plane) and to establish uniqueness; the same is done, explicitly, in [1]. Eliminating the need for this identity was one motivation for this paper.

The main theorem in [16] states the existence and uniqueness of a bounded vorticity bounded velocity solution to the Euler equations that is unique among all such solutions having sublinear growth of the pressure at infinity. What is actually proven in [16], however, is the existence and uniqueness of a bounded vorticity bounded velocity solution to the Euler equations satisfying the identity in (2.2). Another motivation for this paper was to clarify this point by proving the result that Serfati actually stated. This is the content of Theorems 2.9 and 2.10.

The vanishing viscosity limit of the Navier-Stokes equations to the Euler equations has been studied for bounded solutions in [6, 7, 8].

Finally, in the recent paper [9], Gallay obtains an identity for a bounded vorticity bounded velocity vector field, u, that is complementary to the renormalized Biot-Savart law of (1.4). Rather than cutting off the Biot-Savart law he truncates the vorticity then takes the limit as $R \to \infty$. To allow this, he first "tames" the Biot-Savart kernel. He finds that

$$u(x) = u(0) + \lim_{R \to \infty} \int_{B_R} (K(x-y) - K(y))\, \omega(y)\, dy.$$

(For $L^1 \cap L^\infty$ vorticity this identity would follow directly from the Biot-Savart law.) He uses this to (among other things) obtain the linear-in-time growth of the L^∞-norm of the velocity for solutions to the Navier-Stokes equations with a bound that is uniform in small viscosity and hence applies in the limit of zero viscosity to the Euler equations.

This paper is organized as follows:

In Section 2 we define our bounded solutions to the 2D Euler equations and state our main results. We summarize some background facts and definitions in Section 3 that we will use throughout the paper.

Section 4 contains a proof of the renormalized Biot-Savart law, which we use in in Section 5 to characterize bounded solutions for the full plane. The proof of existence and uniqueness is given in Section 6. In Section 7, we establish the properties of the pressure for the full plane. The formula for the pressure gradient in the full plane is the same as that in [17], and is based on the Green's function for the Laplacian. The most delicate estimates, those characterizing the behavior of the pressure itself at infinity, we obtain using a Riesz transform. These estimates are presented in Section 8.

In Section 9, we make a few final comments concerning the nature of the weak solutions we have defined, and discuss our results in relation to the literature.

2. Statement of results

Before we can state our results, we must make several definitions.

For a velocity field, u, the vorticity, $\omega(u) = \operatorname{curl}(u) := \partial_1 u^2 - \partial_2 u^1$.

Let $G(x, y) = (2\pi)^{-1} \log |x - y|$, the fundamental solution to the Laplacian in $\mathbb{R}^2$. Then the Biot-Savart kernel in the full plane is given by

$$K(x) = \nabla^\perp G(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \tag{2.1}$$

where $\nabla^\perp := (-\partial_2, \partial_1)$ and $x^\perp := (-x_2, x_1)$. When ω is a compactly supported, bounded scalar field, we define

$$K[\omega] = K * \omega.$$

Then $K[\omega]$ is the unique, divergence-free vector field vanishing at infinity whose vorticity is ω.

Definition 2.1. We say that a divergence-free vector field, $u \in L^\infty(\mathbb{R}^2)$, with vorticity, $\omega(u) \in L^\infty(\mathbb{R}^2)$ is a *Serfati velocity*. We call the space of all such vector fields, $S = S(\mathbb{R}^2)$, with the norm,

$$\|u\|_S = \|u\|_{L^\infty} + \|\omega(u)\|_{L^\infty}.$$

Definition 2.2. We say that a sequence, (u_n), in $L^\infty(0, T; S)$ *converges locally in* S if for any compact subset, L, of $\mathbb{R}^2$,

$$\|u_n - u\|_{L^\infty([0,T]\times L)} + \|\omega(u) - \omega(u_n)\|_{L^\infty([0,T]\times L)} \to 0.$$

We will use the following definition for solutions in the full plane:

Definition 2.3. Fix $T > 0$. We say that a velocity field, u, lying in $L^\infty(0, T; S) \cap C([0, T] \times \mathbb{R}^2)$ having vorticity, $\omega = \omega(u)$, is a *bounded solution* to the Euler equations without forcing if, on the interval, $[0, T]$, $\partial_t \omega + u \cdot \nabla\omega = 0$ as distributions on $(0, T) \times \mathbb{R}^2$.

Remark 2.4. Because the velocity, u, of Definition 2.3 lies in $L^\infty(0, T; S) \cap C([0, T] \times \mathbb{R}^2)$, it follows from Lemma 3.5 that u has a spatial log-Lipschitz modulus of continuity (MOC) with a uniform bound over $[0, T]$ and thus that it has a unique classical flow map. Moreover, this flow map is measure-preserving and the vorticity is transported by the flow map.

Remark 2.5. The vorticity equation, $\partial_t \omega + u \cdot \nabla\omega = 0$, in Definition 2.3 is not a vorticity formulation, since we do not specify how the velocity field is recovered from the vorticity. Indeed, the key fact we show in this paper is that the membership of $u(t)$ in S forces the recovery of the velocity from the vorticity to take place in the specific manner given by $(1.3)_1$ (more precisely stated in Theorem 2.9). The only freedom is the choice of U_∞. (We can use this observation to define a vorticity formulation, as we explain in Section 9.1.)

Definition 2.6. Let a be a radially symmetric, smooth, compactly supported function with $a = 1$ in a neighborhood of the origin. We will refer to such a function simply as a **radial cutoff function**. For any $R > 0$ we define

$$a_R(\cdot) = a(\cdot/R).$$

Definition 2.7. For v, w vector fields, we define $v * \cdot w = v^i * w^i$. For A, B matrix-valued functions on $\mathbb{R}^2$, we define $A * \cdot B = A^{ij} * B^{ij}$. Here, and throughout this paper, we use the convention that repeated indices are summed over.

Our main results are Lemma 2.8, Theorem 2.9, and Theorem 2.10.

Lemma 2.8. *Assume that u lies in the Serfati space, S, of Definition 2.1. Let $\omega = \omega(u)$ and define*

$$u_R = (a_R K) * \omega.$$

Then $\omega(u_R) \to \omega(u)$ in L^∞ with $\|\omega(u_R) - \omega(u)\|_{L^\infty} \le C \|u\|_{L^\infty} R^{-1}$, and there exists a subsequence, (R_k), $R_k \to \infty$, and a constant vector field, H, such that $u_{R_k} \to u + H$ as $k \to \infty$ uniformly on compact subsets.

Theorem 2.9 (Characterization of solutions)**.** *Suppose that u is a solution to the Euler equations as in Definition 2.3 in the full plane with initial velocity, $u(t = 0) = u^0 \in S$, and initial vorticity, $\omega^0 = \omega(u^0)$. There exists $U_\infty \in (C[0, T])^2$ with $U_\infty(0) = 0$, such that each of the following holds:*
(i) Serfati identity: for $j = 1, 2$,

$$\begin{aligned} u^j(t) - (u^0)^j &= U^j_\infty(t) + (aK^j) * (\omega(t) - \omega^0) \\ &\quad - \int_0^t \left(\nabla\nabla^\perp \left[(1 - a)K^j\right]\right) * \cdot (u \otimes u)(s)\, ds. \end{aligned} \tag{2.2}$$

(ii) Renormalized Biot-Savart law:

$$u(t) - u^0 = U_\infty(t) + \lim_{R\to\infty} (a_R K) * (\omega(t) - \omega^0) \tag{2.3}$$

on $[0, T] \times \mathbb{R}^2$ for all radial cutoff functions, a, as in Definition 2.6. The convergence in (2.3) is locally uniform in S as in Definition 2.2.
(iii) There exists a pressure field $p \in \mathcal{D}'((0, T) \times \mathbb{R}^2)$ with $\nabla p + U'_\infty$ lying in $L^\infty([0, T] \times \mathbb{R}^2)$, such that

$$\partial_t u + u \cdot \nabla u + \nabla p = 0 \tag{2.4}$$

as distributions on $(0, T) \times \mathbb{R}^2$. Here, $\partial_t u - U'_\infty \in L^\infty(0, T; L^r_{loc}(\mathbb{R}^2))$ for all r in $[1, \infty)$. (Note that $U'_\infty \in (\mathcal{D}'((0, T)))^2$.)
(iv) For any radial cutoff function, a, as in Definition 2.3,

$$\begin{aligned} \nabla p(t, x) &= -U'_\infty(t) + \int_{\mathbb{R}^2} a(x - y) K^\perp(x - y) \operatorname{div}\operatorname{div}(u \otimes u)(t, y)\, dy \\ &\quad + \int_{\mathbb{R}^2} (u \otimes u)(t, y) \cdot \nabla_y \nabla_y \left[(1 - a(x - y))K^\perp(x - y)\right] dy. \end{aligned} \tag{2.5}$$

Also, $\|\nabla p(t) + U'_\infty(t)\|_{L^\infty} \le C\|u^0\|^2_S$.
(v) Pressure growth at infinity: The pressure, p, can be chosen so that

$$p = -U'_\infty \cdot x - R(u \otimes u), \tag{2.6}$$

where $R = \Delta^{-1} \operatorname{div} \operatorname{div}$ *is a Riesz transform on* 2×2 *matrix-valued functions on* $\mathbb{R}^2$*. Moreover,*

$$p(t, x) + U_\infty'(t) \cdot x \in L^\infty([0, T]; BMO) \tag{2.7}$$

with

$$p(t, x) = -U_\infty'(t) \cdot x + O(\log |x|), \tag{2.8}$$

Theorem 2.10. *Assume that* $u^0 \in S$*, let* $T > 0$ *be arbitrary, and fix* $U_\infty \in (C[0, T])^2$ *with* $U_\infty(0) = 0$*. There exists a bounded solution,* u*, to the Euler equations as in Definition 2.3, and this solution satisfies (i)-(v) of Theorem 2.9. This solution is unique among bounded all solutions with* $u(0) = u^0$ *that satisfy any one of the following uniqueness criteria:*

(a) *(i) of Theorem 2.9 holds;*
(b) *(ii) of Theorem 2.9 holds;*
(c) *there exists a pressure satisfying (2.4, 2.6) for which (2.7) holds;*
(d) *there exists a pressure satisfying (2.4, 2.6) for which* $\nabla p + U_\infty' \in L^\infty([0, T] \times \mathbb{R}^2)$ *and (2.8) holds.*

Remark 2.11. Radial symmetry of the cutoff function, a, simplifies some of our proofs, so we adopt it, but it is not a necessary assumption.

Theorem 2.9 shows that if one has a bounded solution to the Euler equations then there must be a U_∞ for which the solution has the stated properties. Theorem 2.10 is a kind of converse, which says that if one has a U_∞ there does, in fact, exist a bounded solution to the Euler equations that satisfies one of the properties stated in Theorem 2.9. By the uniqueness in Theorem 2.10 it then follows that the solutions whose existence is ensured by that theorem satisfies all of the properties given in Theorem 2.9.

We begin the proof of Theorem 2.9 in Section 5 by establishing properties *(i)* and *(ii)*, thereby characterizing the velocity for bounded solutions in the full plane. Theorem 2.10, giving the existence of solutions along with uniqueness of such solutions that satisfy (2.2), follows easily from the construction of Serfati solutions in [1] and the transformation in (1.5): this is explained in detail in Section 6. It follows from this uniqueness, then, that any further properties we can establish for the Serfati solutions constructed in [1], modified by (1.5), must hold for our bounded solutions. In Section 7 we establish some such properties; namely, those of the pressure appearing in *(iii)*-*(v)* of Theorem 2.9.

The formula for the pressure gradient in the full plane is the same as that in [17], and is based on the Green's function for the Laplacian. The most delicate estimates, those characterizing the behavior of the pressure itself at infinity, we obtain using Riesz transforms in the full plane. These estimates appear in Section 8.

Remark 2.12. It is possible to obtain results analogous to Lemma 2.8, Theorem 2.9, and Theorem 2.10 for the exterior to a simply connected obstacle; this is the subject of a future work.

3. Background Material

In this section we present definitions and bounds that we will need in the remainder of this paper.

We have the following estimates on K of (2.1):

Proposition 3.1. *We have,*

$$|K(x-y)| \le \frac{C}{|x-y|}. \tag{3.1}$$

Let a be a radial cutoff function. There exists $C > 0$ such that for all $\varepsilon > 0$,

$$\left\| \nabla_y a_\varepsilon(x-y) \otimes \nabla_y K^i(x-y) \right\|_{L^1_y(\mathbb{R}^2)} \le C\varepsilon^{-1}, \tag{3.2}$$

$$\left\| \nabla_y \nabla_y \left[(1 - a_\varepsilon(x-y)) K(x-y) \right] \right\|_{L^1_y(\mathbb{R}^2)} \le C\varepsilon^{-1}. \tag{3.3}$$

Let $U \subseteq \mathbb{R}^2$ have measure $2\pi R^2$ for some $R < \infty$. Then for any p in $[1, 2)$,

$$\|K(x-\cdot)\|^p_{L^p(U)} \le \frac{R^{2-p}}{2-p}. \tag{3.4}$$

Proof. The bound in (3.1) is immediate from (2.1). For the bounds in (3.2-3.4) see [1]. □

Definition 3.2. A nondecreasing continuous function, $\mu\colon [0,\infty) \to [0,\infty)$, is a modulus of continuity (MOC) if $\mu(0) = 0$ and $\mu > 0$ on $(0, \infty)$.

Definition 3.3 is a generalization of Hölder-continuous functions.

Definition 3.3. Let μ be a MOC. Define

$$\begin{aligned} C_\mu = C_\mu(\mathbb{R}^2) = \{ f \in C_b(\mathbb{R}^2) \colon \exists\, c_0 > 0 \text{ s.t. } \forall\, x, y \in \mathbb{R}^2, \\ |f(x) - f(y)| \le c_0 \mu(|x-y|) \} \end{aligned}$$

with

$$\|f\|_{C_\mu} = \|f\|_{L^\infty} + \|f\|_{\dot{C}_\mu},$$

where

$$\|f\|_{\dot{C}_\mu} = \sup_{x \ne y} \frac{|f(x) - f(y)|}{\mu(|x-y|)}.$$

We define Log-Lipschitz functions explicitly by using the MOC,

$$\mu_{LL}(r) = \begin{cases} -r \log r, & \text{if } r \le e^{-1}, \\ e^{-1}, & \text{if } r > e^{-1}, \end{cases} \tag{3.5}$$

setting $LL = C_{\mu_{LL}}$.

Definition 3.4. Given a MOC, μ, we define,

$$S_\mu(x) = \int_0^x \frac{\mu(r)}{r}\, dr.$$

We say that μ is *Dini* if S_μ is finite for some (and hence all) $x > 0$. (Note that when μ is Dini, S_μ is itself a MOC.) A function is *Dini-continuous* if it has a Dini MOC.

Lemma 3.5 gives the MOC for a bounded velocity field and a bound on its gradient.

Lemma 3.5. *Suppose $u \in S$. Then $u \in LL$ with $\|u\|_{LL} \le C \|u\|_S$. Moreover, for any bounded domain $D \subseteq \mathbb{R}^2$ and any $p \in (1, \infty)$,*

$$\|\nabla u\|_{L^p(D)} \le C |D|^{1/p} \frac{p^2}{p-1} \|u\|_S .$$

Proof. See [1]. □

Let $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^2)$ be the space of tempered distributions and $\mathcal{E}' = \mathcal{E}'(\mathbb{R}^2)$ be the subspace of compactly supported tempered distributions. We make frequent use of the following classical result:

Lemma 3.6. *Suppose that $f \in \mathcal{E}'$ and $g \in \mathcal{S}'$. Then $f * g = g * f$ lies in $\mathcal{S}'$ and*

$$D^\alpha (f * g) = D^\alpha f * g = f * D^\alpha g$$

for all multi-indices, α.

The following are two integration-by-parts lemmas for low regularity solutions; the first is a standard fact, the second is Theorem I.1.2 of [20].

Lemma 3.7. *Let U be an open subset of $\mathbb{R}^2$. If f lies in $H_0^1(U)$ and v lies in $(L^2(U))^2$ then $\operatorname{div} v$ lies in $H^{-1}(U)$ and*

$$(\operatorname{div} v, f)_{H^{-1}(U), H_0^1(U)} = -\int_U \nabla f \cdot v.$$

Lemma 3.8. *Let U be an open subset of $\mathbb{R}^2$ with smooth boundary. Let $E(U) = \{u \in (L^2(U))^2 \colon \operatorname{div} u \in L^2(U)\}$ endowed with the norm, $\|u\|_{E(U)} = \|u\| + \|\operatorname{div} u\|$. There exists a continuous trace operator from $E(U)$ to $H^{-1/2}(\partial U)$, which we write as $u \mapsto u \cdot \boldsymbol{n}$, that extends the restriction to the boundary of the normal component of u for continuous vector fields. Assume that f lies in $H^1(U)$ and u lies in $E(U)$. Then*

$$\int_U u \cdot \nabla f + \int_U \operatorname{div} u \, f = (u \cdot \boldsymbol{n}, f)_{H^{-1/2}(\partial U), H^{1/2}(\partial U)},$$

where f is the usual trace operator from $H^1(U)$ to $H^{1/2}(U)$ applied to f.

4. The renormalized Biot-Savart law

In this section we prove Lemma 2.8 after establishing several lemmas. The first of these gives some basic facts regarding the Biot-Savart kernel treated as a tempered distribution.

Lemma 4.1. *We have, $\operatorname{div} K = \operatorname{div}(a_R K) = 0$ and*

$$\omega(a_R K) = \delta + \nabla^\perp a_R \cdot K.$$

Proof. Formally,

$$\begin{aligned}
\operatorname{div} K &= \operatorname{div} \nabla^\perp G = 0, \\
\operatorname{div}(a_R K) &= a_R \operatorname{div} K + \nabla a_R \cdot K = 0, \\
\omega(a_R K) &= -\operatorname{div}(a_R K^\perp) = -\nabla a_R \cdot K^\perp - a_R \operatorname{div} K^\perp \\
&= \nabla^\perp a_R \cdot K + a_R \operatorname{div} \nabla G = \nabla^\perp a_R \cdot K + \delta,
\end{aligned}$$

and these can all be proved to hold as equality of distributions by applying test functions.

Let φ be any test function in $\mathcal{S}(\mathbb{R}^2)$. We will show that $(K, \nabla\varphi) = 0$, from which it follows by de Rham that $\operatorname{div} K = 0$.

Because K is locally integrable,

$$\begin{aligned}
(K, \nabla\varphi) &= \int_{\mathbb{R}^2} K(y) \cdot \nabla\varphi(y)\, dy = \lim_{\delta\to 0} \int_{B_\delta^C} K(y) \cdot \nabla\varphi(y)\, dy \\
&= -\lim_{\delta\to 0} \int_{B_\delta^C} \operatorname{div} K(y)\varphi(y)\, dy + \lim_{\delta\to 0} \int_{\partial B_\delta^C} (K \cdot \boldsymbol{n})\varphi\, dy.
\end{aligned}$$

This vanishes, since $\operatorname{div} K = 0$ away from the origin and $K \cdot \boldsymbol{n} = 0$ on ∂B_δ^C.

Because $a_R K$ is the product of a smooth function and a distribution, the product rule applies, and we have

$$\operatorname{div}(a_R K) = a_R \operatorname{div} K + \nabla a_R \cdot K = 0,$$

a_R being radially symmetric. Also applying the product rule,

$$\begin{aligned}
\omega(a_R K) &= -\operatorname{div}(a_R K^\perp) = -\nabla a_R \cdot K^\perp - a_R \operatorname{div} K^\perp \\
&= \nabla^\perp a_R \cdot K + a_R \operatorname{div} \nabla G = \nabla^\perp a_R \cdot K + \delta,
\end{aligned}$$

using $a_R(0) = 1$ and $\operatorname{div} \nabla G = \Delta G = \delta$ with G as in (2.1).

□

Lemma 4.2. *Let α, β be multi-indices with $|\alpha| \ge 1$ and $|\beta| \ge 0$. Then*

$$\|D^\alpha a_R \otimes D^\beta K\|_{L^1} \le C R^{1-|\alpha|-|\beta|}.$$

Moreover, if $F \in L^\infty(\mathbb{R}^2)$ then

$$\|(D^\alpha a_R \otimes D^\beta K) * F\|_{L^\infty} \le C \|F\|_{L^\infty} R^{1-|\alpha|-|\beta|}.$$

Proof. The L^1-bound follows because $D^\alpha a_R$ is supported on an annulus of inner radius, $c_1 R$, and outer radius, $c_2 R$, for some $0 < c_1 < c_2$, and is bounded by $CR^{-\alpha}$ on this annulus, while $|\partial_\beta K| \le CR^{-\beta-1}$ on this annulus. The bound $(D^\alpha a_R \otimes D^\beta K) * F$ then follows from Young's convolution inequality. □

Lemma 4.3. *For all $f \in \mathcal{E}'$, $v \in (\mathcal{S}')^2$,*

$$\nabla f \mathbin{*\cdot} v = f * \operatorname{div} v,$$

where the $*\cdot$ operator is as in Definition 2.7.

Proof. Using Lemma 3.6,

$$\nabla f \ast\cdot v = \partial_i f * v^i = f * \partial_i v^i = f * \operatorname{div} u.$$

□

Lemma 4.4 allows us to move the curl operator from the velocity field onto the compactly supported distribution, $a_R K$. Even formally, this equality does not follow immediately, and is, in fact, true only when a_R is radially symmetric (see Remark 4.5).

Lemma 4.4. *For any $u \in S$, $(a_R K) * \omega(u) = \omega(a_R K) * u$.*

Proof. We will show that $w := (a_R K) * \omega(u) - \omega(a_R K) * u = 0$. We have,

$$\begin{aligned} w^i &= (a_R K^i) * (\partial_1 u^2 - \partial_2 u^1) - (\partial_1(a_R K^2) - \partial_2(a_R K^1)) * u^i \\ &= \partial_1(a_R K^i) * u^2 - \partial_2(a_R K^i) * u^1 - (\partial_1(a_R K^2) - \partial_2(a_R K^1)) * u^i. \end{aligned}$$

Then,

$$\begin{aligned} w^1 &= \partial_1(a_R K^1) * u^2 - \partial_2(a_R K^1) * u^1 - (\partial_1(a_R K^2) - \partial_2(a_R K^1)) * u^1 \\ &= \partial_1(a_R K^1) * u^2 - \partial_1(a_R K^2) * u^1 \\ &= (\partial_1 a_R K^1) * u^2 - (\partial_1 a_R K^2) * u^1 + (a_R \partial_1 K^1) * u^2 - (a_R \partial_1 K^2) * u^1 \\ &= (\partial_1 a_R K^1) * u^2 - (\partial_1 a_R K^2) * u^1 - (a_R \partial_2 K^2) * u^2 - (a_R \partial_1 K^2) * u^1 \\ &= (\partial_1 a_R K^1) * u^2 - (\partial_1 a_R K^2) * u^1 + (\partial_2 a_R K^2) * u^2 + (\partial_1 a_R K^2) * u^1 \\ &\qquad - \partial_2(a_R K^2) * u^2 - \partial_1(a_R K^2) * u^1 \\ &= (\partial_1 a_R K^1) * u^2 + (\partial_2 a_R K^2) * u^2 - \nabla(a_R K^2) \ast\cdot u \\ &= (\nabla a_R \cdot K) * u^2 = 0, \end{aligned}$$

since $\nabla a_R \cdot K = 0$, a_R being radially symmetric (note that $\nabla a_R \cdot K$ is integrable). In the fourth equality we used $\operatorname{div} K = 0$ from Lemma 4.1, and we applied Lemma 4.3 in the penultimate equality to deduce that $\nabla(a_R K^2) \ast\cdot u = (a_R K^2) * \operatorname{div} u = (a_R K^2) * 0 = 0$. Similarly,

$$w^2 = -(\nabla a_R \cdot K) * u^1 = 0.$$

$$\begin{aligned} w^2 &= \partial_1(a_R K^2) * u^2 - \partial_2(a_R K^2) * u^1 - (\partial_1(a_R K^2) - \partial_2(a_R K^1)) * u^2 \\ &= -\partial_2(a_R K^2) * u^1 + \partial_2(a_R K^1) * u^2 \\ &= -(\partial_2 a_R K^2) * u^1 + (\partial_2 a_R K^1) * u^2 - (a_R \partial_2 K^2) * u^1 + (a_R \partial_2 K^1) * u^2 \\ &= -(\partial_2 a_R K^2) * u^1 + (\partial_2 a_R K^1) * u^2 + (a_R \partial_1 K^1) * u^1 + (a_R \partial_2 K^1) * u^2 \\ &= -(\partial_2 a_R K^2) * u^1 + (\partial_2 a_R K^1) * u^2 - (\partial_1 a_R K^1) * u^1 - (\partial_2 a_R K^1) * u^2 \\ &\qquad + \partial_1(a_R K^1) * u^1 + \partial_2(a_R K^1) * u^2 \\ &= -(\partial_2 a_R K^2) * u^1 - (\partial_1 a_R K^1) * u^1 + \nabla(a_R K^1) \ast\cdot u \\ &= -(\nabla a_R \cdot K) * u^1 = 0. \end{aligned}$$

□

Remark 4.5. The radial symmetry of a was convenient in the proof of Lemma 2.8, but was not essential. Were a not radially symmetric, an application of Lemma 3.6 would give $(\nabla a_R \cdot K) * \omega = (\nabla^\perp(\nabla a_R \cdot K)) *\cdot\, u$. This is $O(R^{-1})$ by Lemma 4.2 (and the product rule), so $\operatorname{div} u_R \to 0$ in $L^\infty(\mathbb{R}^2)$, which yields $\operatorname{div} \overline{u} = 0$. Also, Lemma 4.4 would become $u_R = \omega(a_R K) * u - (\nabla a_R \cdot K) * u^\perp$, but the extra term $(\nabla a_R \cdot K) * u^\perp$ can be handled just as $(\nabla^\perp a_R \cdot K) * u$ is.

Proof of Lemma 2.8. First observe that u_R is well-defined as a tempered distribution by Lemma 3.6, since $a_R K \in \mathcal{E}'$. Also by that lemma and Lemma 4.1,

$$\operatorname{div} u_R = (\operatorname{div}(a_R K)) * \omega = 0 * \omega = 0.$$

Then, from Lemmas 4.1 and 4.4,

$$\begin{aligned} u_R &= \omega(a_R K) * u \\ &= (\delta + \nabla^\perp a_R \cdot K) * u = u + (\nabla^\perp a_R \cdot K) * u. \end{aligned}$$

But, $(\nabla^\perp a_R \cdot K) * u$ is $O(1)$ by Lemma 4.2, so (u_R) is bounded in L^∞.

Since also $\omega((\nabla^\perp a_R \cdot K) * u) = O(R^{-1})$ by Lemma 4.2, we have

$$\omega(u_R) = O(R^{-1}) + \omega(u).$$

We conclude both that $\omega(u_R) \to \omega(u)$ in L^∞ and that (u_R), already bounded in L^∞, is bounded in S.

By Lemma 3.5, then, (u_R) is an equicontinuous family of pointwise bounded functions and hence for any compact subset, L, of $\mathbb{R}^2$ some subsequence of (u_R) converges uniformly on L. A diagonalization argument for increasing L gives a subsequence, (u_{R_k}), that converges uniformly on compact subsets to some $\overline{u}$ in L^∞. At the same time, as shown above, $\omega(u_R) \to \omega(u)$ and $\operatorname{div} u_R = 0$.

Fix a compact subset, L, of $\mathbb{R}^2$ and let $\varphi \in H_0^1(L)$. Then

$$(\omega(u_{R_k}), \varphi) = -(\operatorname{div} u_{R_k}^\perp, \varphi) = (u_{R_k}^\perp, \nabla\varphi) \to (\overline{u}^\perp, \nabla\varphi) = (\omega(\overline{u}), \varphi).$$

But also $(\omega(u_R), \varphi) \to (\omega(u), \varphi)$, so $\omega(\overline{u}) = \omega(u)$ on L and hence on all of $\mathbb{R}^2$, since L was arbitrary. Similarly, $\operatorname{div} \overline{u} = \operatorname{div} u = 0$.

Thus, $\operatorname{div}(u - \overline{u}) = 0$ and $\omega(u - \overline{u}) = 0$. By the identity, $\Delta v = \nabla \operatorname{div} v + \nabla^\perp \omega(v)$, then, $\Delta(u - \overline{u}) = 0$, and we conclude that $\overline{u} = u + H$, where H is an harmonic polynomial. Since u and $\overline{u}$ lie in L^∞, H must be a constant. □

5. Characterization of velocity at infinity

In this section we prove (i) and (ii) of Theorem 2.9 on the characterization of velocity at infinity. The proof rests upon the equivalence between the renormalized Biot-Savart law and the Serfati identity as given in Proposition 5.1, which we first state, returning to its proof following the proof of (i) and (ii) of Theorem 2.9.

Proposition 5.1. *Suppose that u is a solution to the Euler equations in the full plane as in Definition 2.3. If u satisfies (2.2) for some U_∞ then (2.3) holds, the convergence being uniform on compact subsets of $[0,T]\times\mathbb{R}^2$. Conversely, if (2.3) holds for a subsequence for some U_∞, the convergence being pointwise for any fixed $t \in [0,T]$, then u satisfies (2.2). The subsequence is allowed to vary with $t \in [0,T]$.*

Remark 5.2. It follows from Proposition 5.1 that if (2.3) holds for a subsequence, the convergence being pointwise for any fixed $t \in [0,T]$, then the convergence actually holds for the full sequence and is uniform on compact subsets of $[0,T]\times\mathbb{R}^2$.

Proof of Theorem 2.9 (i, ii). Suppose that u is a solution to the Euler equations as in Definition 2.3 and a is any radial cutoff function as in Definition 2.6. Then from Lemma 2.8 there exists a subsequence, (R_k), for which

$$u(t) - u^0 = U_\infty(t) + \lim_{k\to\infty}(a_{R_k}K) * (\omega(t) - \omega^0)$$

for some vector field, $U_\infty(t)$. By Proposition 5.1 and Remark 5.2, the limit then holds for the entire sequence, uniformly on compact subsets of $[0,T]\times\mathbb{R}^2$, both (2.2, 2.3) hold, and $U_\infty \in C([0,T])$. Appealing to Lemma 2.8 once more, we see that the limit in (2.3) holds locally in S (in fact, the vorticities converge in $L^\infty(\mathbb{R}^2)$). By Proposition 5.3, U_∞ is independent of the choice of cutoff function, a.

It then follows from (2.2), the transport of the vorticity by the flow map, the boundedness of the velocity, the absolute continuity of the integral, the continuity of u in $L^\infty([0,T])$, and the continuity of U_∞, that $U_\infty(0) = 0$. $\square$

To prove Proposition 5.1 we must first establish the independence of the Serfati identity on the choice of cutoff function, as given in Proposition 5.3. Its proof rests upon a technical lemma, Lemma 5.4, which we state and prove last.

Proposition 5.3. *Suppose that u is a solution to the Euler equations in the full plane as in Definition 2.3 and that (2.2) holds for one, given cutoff function, a. Then (2.2) holds for any other cutoff function, b.*

Proof. Let $R_a(t,x)$ be the right-hand side of (2.2) for the cutoff function, a, and note that it is always finite for any u in $L^\infty(0,T;S)$. Letting $h(y) = (a(y) - b(y))K^j(y)$, $j = 1$ or 2, h lies in $H^2(\mathbb{R}^2)$ and has compact support, so by Lemma 5.4,

$$\begin{aligned}&R_b(t,x) - R_a(t,x)\\ &\qquad = -h * (\omega(t) - \omega^0)(x) - \int_0^t (\nabla\nabla^\perp h) *\cdot (u\otimes u)(s,x)\,ds = 0.\end{aligned}$$

$\square$

Proof of Proposition 5.1. Assume that (2.2) holds. Because the vorticity is transported by the flow map and the velocity is continuous in time and space, both integrals in (2.2) are continuous as functions of t and x. Therefore, it must be that $U_\infty \in C([0,T])$.

By Proposition 5.3, (2.2) holds for a_R in place of a for all $R > 0$. Taking the limit as $R \to \infty$ and applying (3.3) gives (2.3), the convergence being uniform on compact subsets of $[0,T] \times \mathbb{R}^2$.

Now assume that (2.3) holds for a subsequence, (R_k), with the convergence being pointwise for any fixed $t \in [0,T]$. Because t is fixed in the argument that follows, it does not matter whether the subsequence varies with time. Fixing x in $\mathbb{R}^2$ and letting $h(y) = (a_{R_k} - a)(x-y)K^j(x-y)$, $j = 1$ or 2, Lemma 5.4 gives

$$\begin{aligned}
&((a_{R_k} - a)K^j) * (\omega(t) - \omega^0) \\
&\qquad = \int_0^t \nabla\nabla^\perp \left[(a_{R_k} - a)K^j\right] *\cdot (u \otimes u)(s)\, ds.
\end{aligned} \tag{5.1}$$

Because of (2.3), as $k \to \infty$, the left hand side of (5.1) converges to

$$u^j(t,x) - (u^0)^j(x) - U_\infty(t) - (aK^j) * (\omega(t) - \omega^0).$$

The right-hand side of (5.1) can be written,

$$\int_0^t \nabla\nabla^\perp \left[(1-a)K^j\right] *\cdot (u \otimes u)(s)\, ds \\
- \int_0^t \nabla\nabla^\perp \left[(1-a_{R_k})K^j\right] *\cdot (u \otimes u)(s)\, ds.$$

Applying (3.3) with Young's convolution inequality to the second term above we see that it vanishes as $R_k \to \infty$ (here, we need only that $u \in L^\infty([0,T] \times \mathbb{R}^2)$. Taking the limit as $k \to \infty$, then, it follows that (2.2) holds and hence also, as observed above, $U_\infty \in C([0,T])$. □

Formally, Lemma 5.4 follows from several integrations by parts, but we must take some care to do these integrations in the face of the fairly minimal time regularity of the vorticity for our weak solutions. (The convolutions in space will all be of a compactly supported distribution with a tempered distribution, and so represent no difficulties.)

Lemma 5.4. *Let $h \in H^2(\mathbb{R}^2)$ have compact support. Assume that u is a bounded solution to the Euler equations as in Definition 2.3. Then*

$$h * (\omega(t) - \omega^0) = -\int_0^t (\nabla\nabla^\perp h) *\cdot (u \otimes u)(s)\, ds. \tag{5.2}$$

Proof. Note that the compact support of h gives the finiteness of both convolutions in (5.2). Define, for all ε in $(0, 1/2)$,

$$h_\varepsilon(s,x) = \phi_\varepsilon(s)h(x),$$

where ϕ_ε lies in $C_C^\infty((0,t))$ and is chosen so that, $\phi_\varepsilon = 1$ on $[\varepsilon, t-\varepsilon]$, $\phi_\varepsilon \geq 0$, and

$$\phi_\varepsilon'(\cdot) \to \delta(\cdot) - \delta(t - \cdot) \text{ as } \varepsilon \to 0^+,$$

the convergence being as Radon measures on $[0,T]$. We note, then, that h_ε lies in $H_0^1((0,t)\times\mathbb{R}^2)$ with compact support in $(0,t)\times\mathbb{R}^2$.

Fix x in $\mathbb{R}^2$ and let B be an open ball in $(0,t)\times\mathbb{R}^2$ sufficiently large to contain $\operatorname{supp} h_\varepsilon(x-\cdot)$.

Now, $\nabla u \in L^\infty(0,T; L^2_{loc}(\mathbb{R}^2))$ since $\omega \in L^\infty([0,T]\times\mathbb{R}^2)$, so $u\cdot\nabla u \in L^\infty(0,T; L^2_{loc}(\mathbb{R}^2))$. Thus, $\partial_t\omega = -u\cdot\nabla\omega = -\operatorname{curl}(u\cdot\nabla u) = \operatorname{div}((u\cdot\nabla u)^\perp)$ lies in $L^\infty(0,T; H^{-1}_{loc}(\mathbb{R}^2))$ and hence in $L^\infty(0,T; H^{-1}(B))$. Therefore, we have sufficient regularity to apply Lemma 3.7 to obtain,

$$\begin{aligned}
(\partial_t\omega, h_\varepsilon(x-\cdot))_{H^{-1}(B),H_0^1(B)} &= (\operatorname{div}((u\cdot\nabla u)^\perp), h_\varepsilon(x-\cdot))_{H^{-1}(B),H_0^1(B)} \\
&= -\int_0^t\int_{\mathbb{R}^2} (u\cdot\nabla u)^\perp(t,y)\cdot\nabla h_\varepsilon(x-y))\,dy\,ds \\
&= \int_0^t\int_{\mathbb{R}^2} (u\cdot\nabla u)(t,y)\cdot\nabla^\perp h_\varepsilon(x-y)\,dy\,ds.
\end{aligned}$$

Using the vector identity, $(u\cdot\nabla u)\cdot V = u\cdot\nabla(V\cdot u) - (u\cdot\nabla V)\cdot u$ with $V = \nabla^\perp h_\varepsilon(x-\cdot)$ gives

$$\begin{aligned}
\int_{\mathbb{R}^2} (u\cdot\nabla u)(t,y)\cdot\nabla^\perp h_\varepsilon(x-y)\,dy &= \int_{\mathbb{R}^2}(u\cdot\nabla u)\cdot V \\
&= \int_{\mathbb{R}^2} u\cdot\nabla(V\cdot u) - \int_{\mathbb{R}^2}(u\cdot\nabla V)\cdot u = -\int_{\mathbb{R}^2}(u\cdot\nabla V)\cdot u.
\end{aligned} \tag{5.3}$$

The one integral vanished because $\operatorname{div} u = 0$ and $\nabla(V\cdot u)\in H^1(\mathbb{R}^2)$ with compact support. We conclude from this that

$$\begin{aligned}
&(\partial_t\omega, h_\varepsilon(x-\cdot))_{H^{-1}(B),H_0^1(B)} \\
&\qquad = -\int_0^t\int_{\mathbb{R}^2}\Big((u(s,y)\cdot\nabla_y)\nabla_y^\perp h_\varepsilon(x-y)\Big)\cdot u(s,y)\,dy\,ds \\
&\qquad = -\int_0^t\int_{\mathbb{R}^2}(\nabla_y\nabla_y^\perp h_\varepsilon(x-y))\cdot(u\otimes u)(s,y)\,dy\,ds \\
&\qquad \to -\int_0^t(\nabla\nabla^\perp h) *\cdot (u\otimes u)(s,x)\,ds
\end{aligned} \tag{5.4}$$

as $\varepsilon\to 0^+$ by the dominated convergence theorem.

With x still fixed, let

$$f(s) = \int_{\mathbb{R}^2} h(x-y)\omega(s,y)\,dy.$$

Vorticity is transported by the flow map (as in Definition 2.3) and u is bounded on $[0,t]\times\mathbb{R}^2$, so f is continuous on $[0,t]$. Thus,

$$\begin{aligned}
&(\partial_t\omega, h_\varepsilon(\cdot, x-\cdot))_{H^{-1}(B),H^1_0(B)}\\
&\qquad = -(\omega, \partial_t h_\varepsilon(\cdot, x-\cdot))_{H^{-1}(B),H^1_0(B)}\\
&\qquad = -(\omega, \phi'_\varepsilon h(x-\cdot))_{L^2(B)},\\
&\qquad = -\int_0^t\int_{\mathbb{R}^2}\phi'_\varepsilon(s)h(x-y)\omega(s,y)\,dy\,ds\\
&\qquad = -\int_0^t\phi'_\varepsilon(s)\int_{\mathbb{R}^2}h(x-y)\omega(s,y)\,dy\,ds\\
&\qquad = -\int_0^t\phi'_\varepsilon(s)f(s)ds \to f(t)-f(0) \text{ as } \varepsilon\to 0^+\\
&\qquad = \int_{\mathbb{R}^2}h(x-y)(\omega(t,y)-\omega^0(y))\,dy.
\end{aligned} \tag{5.5}$$

The identity in (5.2) follows from (5.4, 5.5). □

6. Existence and uniqueness

Our proof of Theorem 2.10 begins with the following lemma:

Lemma 6.1. *Let (u,p) and $(\overline{u},\overline{p})$ be related as in the transformation, (1.5). Then (u,p) satisfy (1.1) if and only if $(\overline{u},\overline{p})$ satisfy (1.1). Moreover, u is a bounded solution to the Euler equations as in Definition 2.3 if and only if $\overline{u}$ is such a solution.*

Proof. Applying the chain rule gives,

$$\begin{aligned}
\partial_t\overline{u}(t,x) &= \partial_t u(t,\overline{x}) + U_\infty(t)\cdot\nabla u(t,\overline{x}) - U'_\infty(t),\\
\nabla\overline{u}(t,x) &= \nabla u(t,\overline{x}),\\
\nabla\overline{p}(t,x) &= \nabla p(t,x) + U'_\infty(t),\\
\operatorname{div}\overline{u}(t,x) &= \operatorname{div} u(t,\overline{x}),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\partial_t\overline{u}(t,x) + \overline{u}(t,x)\cdot\nabla\overline{u}(t,x) + \nabla\overline{p}(t,x)\\
&\qquad = \partial_t u(t,\overline{x}) + u(t,\overline{x})\cdot\nabla u(t,\overline{x}) + \nabla p(t,\overline{x}).
\end{aligned}$$

Thus, (u,p) satisfies (1.1) if and only if $(\overline{u},\overline{p})$ satisfies (1.1) (since $U_\infty(0) = 0$).

Let $\overline{\omega} = \operatorname{curl}\overline{u}$. Then the chain rule gives

$$\begin{aligned}
\overline{\omega}(t,x) &= \omega(t,\overline{x}),\\
\partial_t\overline{\omega}(t,x) &= \partial_t\omega(t,\overline{x}) + \partial_t\overline{x}\cdot\nabla\omega(t,\overline{x})\\
&= \partial_t\omega(t,\overline{x}) + U_\infty(t)\cdot\nabla\omega(t,\overline{x}),\\
\nabla\overline{\omega}(t,x) &= \nabla\omega(t,\overline{x}),
\end{aligned}$$

from which it follows that

$$\partial_t\overline{\omega}(t,x) + \overline{u}(t,x)\cdot\nabla\overline{\omega}(t,x) = \partial_t\omega(t,\overline{x}) + u(t,\overline{x})\cdot\nabla\omega(t,\overline{x}).$$

Hence, the vorticity equation of the Euler equations is satisfied in Definition 2.3 for u if and only if it is satisfied for $\overline{u}$. $\square$

Proof of Theorem 2.10. Assume that $u^0 \in S$, let $T > 0$ be arbitrary, and fix $U_\infty \in (C[0, T])^2$ with $U_\infty(0) = 0$. Let $\overline{u}^0 = u^0 - U_\infty(0) = u^0$, and let $\overline{u}$ be the Serfati solution with initial velocity $\overline{u}^0$ constructed in [1]. Then, as shown in [1], $\overline{u}$ is the unique bounded solution satisfying *(i)* of Theorem 2.9 with $U_\infty \equiv 0$. As we saw in Section 5, *(ii)* is equivalent to *(i)*, and so also holds. Making the inverse change of variables from that in (1.5) then yields a bounded solution, (u, p), satisfying *(i)* and *(ii)* with the original U_∞. This also gives uniqueness criteria (a) and (b).

That *(iii)*-*(v)* hold for (u, p) will be shown when we establish the properties of the pressure in Section 7.

Uniqueness criteria (c) is proved, for $U_\infty \equiv 0$, in [19], and it can also be adapted to a nonzero U_∞ using the change of variables in (1.5). Finally, we observe that uniqueness criteria (d) immediately implies (c). $\square$

Remark 6.2. The solution, $\overline{u}$, constructed in [1] (and hence, by uniqueness, any such solution) also has the property that

$$\|\overline{u}(t)\|_{L^\infty} \le e^{C(1+\|\omega^0\|_{L^\infty})t}\|u^0\|_{L^\infty}.$$

(In [9] this estimate is improved to be linear in time.) Also, $\|\omega(\overline{u})(t)\|_{L^\infty} = \|\omega^0\|_{L^\infty}$, since vorticity is transported by the flow map. Hence,

$$\|\overline{u}(t)\|_S \le e^{C(1+\|\omega^0\|_{L^\infty})t}\|u^0\|_S.$$

Then, since $\|u(t)\|_S = \|\overline{u}(t) - U_\infty(t)\|_S \le \|\overline{u}(t)\|_S + \|U_\infty(t)\|$, we have

$$\|u(t)\|_S \le C_S(t)\|u^0\|_S + \|U_\infty(t)\|, \text{ where } C_S(t) = e^{C(1+\|\omega^0\|_{L^\infty})t}. \tag{6.1}$$

The convenient transformation in (1.5) allowed us to simply use the existence and uniqueness theorem of [1], avoiding the need to modify its proof to accommodate $U_\infty \not\equiv 0$. To establish the properties of the pressure in Theorem 2.9, however, we need the approximate sequence of smooth velocities, (u_n), used in [1] to obtain existence of a solution. Adjusting the sequence in [1] to accommodate U_∞ by employing a sequence, (U_∞^n), converging to U_∞ leads to a sequence, (u_n), of approximate classical solutions with the following properties:

$$\begin{aligned}
&(u_n) \text{ is bounded in } C([0, T] \times S),\\
&u_n \to u \text{ uniformly on compact subsets of } [0, T] \times \mathbb{R}^2,\\
&\omega(u_n) \to \omega(u) \text{ in } L^p_{loc}(\mathbb{R}^2) \text{ for all } p \text{ in } [1, \infty),\\
&u_n(t, x) = U_\infty^n(t) + O(|x|^{-1}),\\
&U_\infty^n \to U_\infty \text{ in } C([0, T]),\\
&(U_\infty^n)' \to U_\infty' \text{ in } \mathcal{D}'((0, T)).
\end{aligned} \tag{6.2}$$

We will use these properties in Section 7.

7. The pressure

In this section, we characterize the pressure for solutions to the 2D Euler equations in the full plane as in $(1.3)_{2,3}$, stated more precisely as properties *(iii)-(v)* of Theorem 2.9.

To understand the difficulties in characterizing the asymptotic behavior of the pressure at infinity, consider first the simpler case of a smooth solution, u, to the Euler equations having compactly supported vorticity with u vanishing at infinity. In such a case, u decays like $C\,|x|^{-1}$ at infinity, while ∇u decays like $C\,|x|^{-2}$ (as in Lemma 7.5).

Taking the divergence of $\partial_t u + u \cdot \nabla u + \nabla p = 0$, we see that p is a solution to $\Delta p = -\operatorname{div}(u \cdot \nabla u) = -\operatorname{div}\operatorname{div}(u \otimes u)$. A particular solution is given by $q = R(u \otimes u)$ for the (multiple) Riesz transform, $R = -\Delta^{-1} \operatorname{div}\operatorname{div}$. Any other solution differs from q by an harmonic polynomial, $h(t)$, so $p = h + q$.

The decay of u gives $u \otimes u \in L^r(\mathbb{R}^2)$ for all $r \in (1, \infty]$. By the Calderón-Zygmund theory, then, $q \in L^r(\mathbb{R}^2)$ for all $r \in (1, \infty)$, so it decays at infinity. Moreover, $\nabla q = T(u \cdot \nabla u)$, where $T = -\Delta^{-1} \nabla \operatorname{div}$ is also a singular integral operator of Calderón-Zygmund type. From the decay of $u \cdot \nabla u$ follows the decay of ∇q at infinity. Then the decay, after integrating in time, of $\partial_t u + u \cdot \nabla u$ at infinity forces h to be constant in space. We conclude that there exists a unique pressure decaying at infinity.

Now let u be a bounded solution to the Euler equations of Definition 2.3. We can still obtain a particular solution, $q = R(u \otimes u)$, to $\Delta p = -\operatorname{div}\operatorname{div}(u \otimes u)$ using the above argument because R maps L^∞ into BMO, and $u \otimes u \in L^\infty$. A bound on the growth of q at infinity could also be obtained formally by applying Proposition 7.2 (this lemma is at the heart of the matter), and rigorously by making a simple approximation argument. Then, arguing as above, we can conclude that *if* a valid pressure exists then it differs from q by an harmonic polynomial, h.

To determine, h, however, we would need to understand the behavior at infinity of $\partial_t u + u \cdot \nabla u$ (at least integrated over time) to obtain a pressure $p = q + h$ satisfying $\partial_t u + u \cdot \nabla u + \nabla p = 0$. But even the behavior of u at infinity is defined only in the weak sense of $(1.3)_1$; it appears to be impossible to say anything useful about the behavior of $\partial_t u + u \cdot \nabla u$ at infinity.

These difficulties naturally lead us to the idea of using an approximate sequence of vector fields, (u_n), decaying sufficiently rapidly at infinity and converging in an appropriate sense to u. We could construct such a sequence in an ad hoc manner, but we already have such a sequence at hand: the sequence of approximate solutions with the properties given in (6.2). This sequence has the virtue that the approach we described above for obtaining a pressure applies to it (after making the transformation in (1.5)), so there exists a corresponding sequence of pressures, (p_n), for which $\partial_t u_n + u_n \cdot \nabla u_n + \nabla p_n = 0$. We will show that this sequence of pressures converges to our desired pressure.

Our proof of *(iii)*-*(v)* of Theorem 2.9 begins by proving Propositions 7.1 through 7.3, which establish properties of the pressure for the approximate solutions, (u_n), of (6.2). Once we establish these properties, it will remain only to make an approximation argument to establish the existence of a pressure, p, for the velocity, u, having the same properties as the approximate sequence of pressures.

Our first proposition provides an explicit expression for the pressure, p_n:

Proposition 7.1. *Let $G(x) = (2\pi)^{-1} \log|x|$, the fundamental solution to the Laplacian in $\mathbb{R}^2$. Let*

$$\begin{aligned} q_n(t,x) &= a_n(t) - G * \operatorname{div}\operatorname{div}(u_n(t) \otimes u_n(t))(x), \\ p_n(t,x) &= -(U_\infty^n)'(t) \cdot x + q_n(t,x), \end{aligned} \tag{7.1}$$

where $a_n(t)$ is chosen so that $p_n(t,0) = q_n(t,0) = 0$ for all t. Then $\partial_t u_n + u_n \cdot \nabla u_n + \nabla p_n = 0$.

Proof. This result for $U_\infty^n \equiv 0$ is classical (the argument being that given at the beginning of this section). For nonzero U_∞^n, we simply use the transformation in (1.5) and apply the first part of Lemma 6.1. $\square$

Our second proposition bounds the growth of p_n (less the harmonic part) at infinity:

Proposition 7.2. *Let q_n be given by $(7.1)_1$. Then,*

$$|q_n(t,x)| \le C C_S(t) \|u^0\|_S^2 \log(e + |x|)$$

for some absolute constant C (in particular, independent of n), where $C_S(t)$ is given in (6.1). Also, q_n has a bound on its log-Lipschitz norm uniform over $[0,T]$ that is independent of n.

Proof. We can write $q_n = a_n(t) - R\,h_n$, where $h_n = u_n \otimes u_n$ and $R = \Delta^{-1} \operatorname{div}\operatorname{div}$ is a Riesz transform. Here, $\Delta^{-1} f = -\mathcal{F}^{-1}(|\cdot|^2 \widehat{f})$, $\mathcal{F}^{-1}$ being the inverse Fourier transform. Observe that $h_n \in LL$ with $\|h_n(t)\|_{LL} \le C\,\|u(t)\|_S^2 \le C_S(t)^2 \|u^0\|_S^2$ by Lemma 3.5 and (6.1). The result then follows from Lemma 8.2, which we prove in the next section. $\square$

Our third proposition give an expression for ∇p_n analogous to (2.5) and shows that it is bounded:

Proposition 7.3. *The identity,*

$$\begin{aligned} \nabla p_n(x) &= -(U_\infty^n)' \\ &\quad + \int_{\mathbb{R}^2} a(x-y) K^\perp(x-y) \operatorname{div}\operatorname{div}(u_n \otimes u_n)(y)\,dy \\ &\quad + \int_{\mathbb{R}^2} (u_n \otimes u_n)(y) \cdot \nabla_y \nabla_y \left[(1 - a(x-y)) K^\perp(x-y) \right] dy, \end{aligned} \tag{7.2}$$

holds independently of the choice of cutoff function, and $\nabla p_n + (U_\infty^n)'$ is bounded uniformly in $L^\infty([0,T] \times \mathbb{R}^2)$.

Proof. Taking the gradient of p_n as given in (7.1), we have

$$\nabla p_n(t,x) = -V_n(t) - \int_{\mathbb{R}^2} \nabla_x G(x-y) \operatorname{div}(u_n \cdot \nabla u_n)(t,y)\, dy,$$

where $V_n = (U^n_\infty)'$.

For $i = 1, 2$ let $j = 2, 1$. Then since $-\nabla_x G(x-y) = K^\perp(x-y)$, we can write

$$(-1)^i \partial_i p_n(x) + (-1)^i V_n^i = \int_{\mathbb{R}^2} K^j(x-y) \operatorname{div}(u_n \cdot \nabla u_n)(y)\, dy.$$

Here, we suppress the time variable to streamline notation. Applying a cutoff and integrating by parts,

$$\begin{aligned}
&(-1)^i \partial_i p_n(x) + (-1)^i V_n^i \\
&\qquad = \int_{\mathbb{R}^2} a(x-y) K^j(x-y) \operatorname{div}(u_n \cdot \nabla u_n)(y)\, dy \\
&\qquad\qquad + \int_{\mathbb{R}^2} (1 - a(x-y)) K^j(x-y) \operatorname{div}(u_n \cdot \nabla u_n)(y)\, dy \\
&\qquad = \int_{\mathbb{R}^2} a(x-y) K^j(x-y) \operatorname{div}(u_n \cdot \nabla u_n)(y)\, dy \\
&\qquad\qquad - \int_{\mathbb{R}^2} (u_n \cdot \nabla u_n)(y) \cdot \nabla \left[(1 - a(x-y)) K^j(x-y)\right]\, dy.
\end{aligned}$$

Integrating as in (5.3) gives

$$\begin{aligned}
&\partial_i p_n(x) + V_n \\
&\quad = (-1)^i \int_{\mathbb{R}^2} a(x-y) K^j(x-y) \operatorname{div}(u_n \cdot \nabla u_n)(y)\, dy \\
&\qquad + (-1)^i \int_{\mathbb{R}^2} (u_n(y) \cdot \nabla_y) \nabla_y \left[(1 - a(x-y)) K^j(x-y)\right] \cdot u_n(y)\, dy,
\end{aligned}$$

which we can write more succinctly as (7.2).

Letting q be Hölder conjugate to p with p in $(1,2)$, we conclude, since $\operatorname{div}(u_n \cdot \nabla u_n) = \nabla u_n \cdot (\nabla u_n)^T$, that

$$\begin{aligned}
\left\| \partial_i p_n + (U^n_\infty)' \right\|_{L^\infty} &\le \|aK\|_{L^p} \|\nabla u_n\|^2_{L^{2q}(\operatorname{supp} a(x-\cdot))} \\
&\qquad + \left\| \nabla_y \nabla_y \left[(1-a) K^j\right] \right\|_{L^1_y} \|u_n\|^2_{L^\infty}.
\end{aligned}$$

But by Lemma 3.5, $\|\nabla u_n\|_{L^{2q}(\operatorname{supp} a(x-\cdot))} \le C \|u_n^0\|_S \le C \|u^0\|_S$. Given the uniform bound on u_n in S it follows from (3.3, 3.4) that $\nabla p_n + (U^n_\infty)'$ lies in $L^\infty([0,T] \times \mathbb{R}^2)$ with a bound that is independent of n.

It is easy to verify that the expression in (7.2) is independent of the choice of cutoff function, a, by subtracting the expression for two different cutoffs then undoing the integrations by parts. (That (2.5) is independent of the choice of cutoff function follows the same way.) $\square$

Proof of (iii)-(v) of Theorem 2.9.
Recall that the sequence (u_n) has the properties in (6.2). Let p_n and q_n be as in Proposition 7.1. By Proposition 7.3, (q_n) is an equicontinuous family on $[0,T]\times\mathbb{R}^2$, so it follows, via Arzela-Ascoli and a simple diagonalization argument applied to an increasing sequence of compact subsets of $\mathbb{R}^2$, that a subsequence of (q_n), which we relabel to use the same indices, converges uniformly on compact subsets, and hence as distributions, to some scalar field, $\overline{q}$. Letting $\overline{p}=-U'_\infty\cdot x+\overline{q}$, it follows that $p_n\to\overline{p}$ in $\mathcal{D}'((0,T)\times\mathbb{R}^2)$ and also that $\overline{p}(t,0)=0$ for all t.

From $(6.2)_{1,2,3}$ it follows that $\partial_t u_n\to\partial_t u$ and $u_n\cdot\nabla u_n\to u\cdot\nabla u$ in $\mathcal{D}'((0,T)\times\mathbb{R}^2)$. But $\nabla p_n\to\nabla\overline{p}$ in $\mathcal{D}'((0,T)\times\mathbb{R}^2)$ and by Proposition 7.1, $\partial_t u_n+u_n\cdot\nabla u_n+\nabla p_n=0$, so $\partial_t u+u\cdot\nabla u+\nabla\overline{p}=0$. Thus, $\overline{p}$ is a valid pressure field, so we can use $p=\overline{p}$.

Because $p_n\to p$ uniformly on compact subsets, (2.7) holds and the bound on $p_n+(U^n_\infty)'$ in Proposition 7.2 yields (2.8). That (2.6) holds follows from Theorem 2 item (1) of [13].

We complete the proof by establishing that (2.5) holds for p and that $\nabla p+U'_\infty\in L^\infty([0,T]\times\mathbb{R}^2)$.

Let Π be the expression on the right-hand side of (2.5). We will show that $\nabla p_n+U'_n\to\Pi+U'$ in $L^\infty([0,T]\times\mathbb{R}^2)$ and hence $\nabla p_n\to\Pi$ in $\mathcal{D}'((0,T)\times\mathbb{R}^2)$. But we already know that $p_n\to p$ in $\mathcal{D}'((0,T)\times\mathbb{R}^2)$ so $\nabla p_n\to\nabla p$ in $\mathcal{D}'((0,T)\times\mathbb{R}^2)$. We can then conclude that $\Pi=\nabla p$, that (2.5) holds, and that $\nabla p+U'\in L^\infty([0,T]\times\mathbb{R}^2)$.

We now show that $\nabla p_n+U'_n\to\Pi+U'$ in $L^\infty([0,T]\times\mathbb{R}^2)$.

We write (7.2) with a replaced by a_ε, where ε is to be determined:

$$\begin{aligned}\nabla p_n(t,x)&=-(U^n_\infty)'(t)+\int_{\mathbb{R}^2}a_\varepsilon(x-y)K^\perp(x-y)\operatorname{div}\operatorname{div}(u_n\otimes u_n)(t,y)\,dy\\&\quad+\int_{\mathbb{R}^2}(u_n\otimes u_n)(t,y)\cdot\nabla_y\nabla_y\left[(1-a_\varepsilon(x-y))K^\perp(x-y)\right]\,dy\\&=:-(U^n_\infty)'(t)+I^n_1(\varepsilon)+I^n_2(\varepsilon).\end{aligned}$$

The value of ∇p_n is independent of our choice of ε, since, by Proposition 7.3, it is independent of the cutoff function a_ε. Let $I_1(\varepsilon)$, $I_2(\varepsilon)$ be the corresponding integrals on the right-hand side of (2.5).

Let $\delta>0$, fix p in $(1,2)$, and let q be Hölder conjugate to p. By Lemma 3.5,

$$\|\nabla u\|_{L^{2q}(\operatorname{supp}a_\varepsilon(x-\cdot))}\le C\varepsilon^{\frac{1}{q}}\|u^0\|_S\le C\varepsilon^{\frac{1}{q}}.$$

Because $\operatorname{div}\operatorname{div}(u\otimes u)=\nabla u\cdot(\nabla u)^T$, this bound gives

$$\|\operatorname{div}\operatorname{div}(u\otimes u)\|_{L^q(\operatorname{supp}a_\varepsilon(x-\cdot))}\le C\varepsilon^{\frac{2}{q}}.$$

Since $|K(x)|=C\,|x|^{-1}$, Hölder's inequality gives

$$\|I_1(\varepsilon)\|_{L^\infty}\le C\varepsilon^{\frac{2}{p}-1+\frac{2}{q}}=C\varepsilon$$

and, similarly, $\|I_1^n(\varepsilon)\|_{L^\infty} \le C\varepsilon$ uniformly for all n. Choose $\varepsilon = \delta/(3C)$ so that $C\varepsilon < \delta/3$. Because $u_n \to u$ uniformly on compact subsets of $([0,T] \times \mathbb{R}^2)$, there exists $N > 0$ such that $n > N \implies \|I_2(\varepsilon) - I_2^n(\varepsilon)\|_{L^\infty} < \delta/3$. (We also use the uniform boundedness of (u_n) to control the tails of the integrals in $I_2(\varepsilon)$, $I_2^n(\varepsilon)$.) Since the value of ∇p_n is independent of ε, this shows that for all $n > N$,

$$\begin{aligned}&\left\|\nabla p_n + (U_\infty^n)' - \Pi - U'\right\|_{L^\infty} \\ &\qquad \le \|I_1(\varepsilon)\|_{L^\infty} + \|I_1^n(\varepsilon)\|_{L^\infty} + \|I_2^n(\varepsilon) - I_2(\varepsilon)\|_{L^\infty} < \delta.\end{aligned}$$

These bounds are uniform in time and in space; hence, $\nabla p_n + (U_\infty^n)' \to \Pi + (U_\infty)'$ in $L^\infty([0,T] \times \mathbb{R}^2)$. Thus, $\nabla p_n \to \Pi$ in $\mathcal{D}'((0,T) \times \mathbb{R}^2)$, since $(U_\infty^n)' \to U'$ in $\mathcal{D}'((0,T))$.

We now have that (2.5-2.8) hold, $\nabla p + U' \in L^\infty([0,T] \times \mathbb{R}^2)$, and $\partial_t u + u \cdot \nabla u + \nabla p = 0$, which completes the proof. □

Remark 7.4. The log-Lipschitz MOC that we obtained in Proposition 7.2 is a side effect of the manner of proof: it is not as strong as the Lipschitz MOC we obtain in Proposition 7.3, though that proposition does not establish decay of p_n.

Lemma 7.5. *For any n there exists a constant, $C > 0$, such that*

$$\begin{aligned}|u_n(\cdot, x) - U_\infty(\cdot)|_{L^\infty([0,T])} &\le \frac{C}{(1 + |x|)}, \\ |\nabla u_n(\cdot, x)|_{L^\infty([0,T])} &\le \frac{C}{(1 + |x|)^2}.\end{aligned}$$

Proof. Because ω_n is compactly supported there is some $R > 0$ such that $\operatorname{supp} \omega_n \subseteq B_R(0)$. Let $|x| > 2R$. Then because u_n is smooth, we have

$$\nabla u_n(x) = (\nabla K) * \omega_n(x) = \int_{B_R(0)} \nabla_x K(x - y) \omega_n(y) \, dy,$$

noting that the compact support of ω eliminates the singularity in $\nabla_x K(x - y)$. But for all $y \in B_R(0)$,

$$|\nabla_x K(x - y)| \le \frac{1}{2\pi(|x| - R)^2} \le \frac{1}{2\pi(|x|/2)^2} \le \frac{2}{\pi |x|^2}$$

so

$$|\nabla u_n(x)| \le \frac{2}{\pi |x|^2} \int_{B_R(0)} |\omega_n(y)| \, dy = \frac{2}{\pi |x|^2} \|\omega_n\|_{L^1}.$$

Since u_n is smooth, ∇u_n is bounded on $B_{2R}(0)$. The bound on ∇u_n follows. The bound on u_n is obtained similarly. □

8. The Poisson problem

In Section 7, we needed to solve the Poisson problem to obtain the pressure in the full plane, our interest being in obtaining the asymptotic behavior of the pressure at infinity. Fortunately, a tool, Lemma 8.1, for obtaining the MOC of the pressure expressed in terms of a Riesz transform exists in the literature, and we can use it to obtain this asymptotic behavior. As applied in Section 7, we do this for the sequence of approximating solutions, which have sufficient decay at infinity so that the Riesz transforms exist in the classical sense of principal values of singular integrals.

Lemma 8.1. *Let R be any Riesz transform in $\mathbb{R}^2$. Suppose that h lying in $L^p(\mathbb{R}^2)$ for some p in $[1,\infty)$ has a concave Dini MOC, μ, as in Definition 3.4. Then Rh has a MOC, ν, given by (see Definition 3.4)*

$$\nu(r) = C\left(S_\mu(r) + r\int_r^\infty \frac{\mu(s)}{s^2}\,ds\right) \tag{8.1}$$

for some absolute constant, C. (Note that this MOC holds for all $r > 0$.)

Proof. This type of bound in dimension higher than one appears to have been first proven by Charles Burch in [3] for a bounded domain (though the MOC he obtains applies only away from the boundary and r must be sufficiently small). It is proved in the whole plane in [14]. □

The following corollary of Lemma 8.1 (though not its proof) is inspired by Lemma 2 of [17].

Lemma 8.2. *Let R be a Riesz transform and assume that h is a tensor field in $LL(\mathbb{R}^2)\cap L^p(\mathbb{R}^2)$ for some p in $[1,\infty)$. Let $q = Rh$. Then q is uniformly continuous with the MOC, $\nu(s) = C\left\|h\right\|_{LL} s(\log s)^2$, for all sufficiently small $s > 0$, and $|q(x) - q(0)| \le C\left\|h\right\|_{LL}\log(e + |x|)$, for some $C > 0$.*

Proof. Referring to (3.5), since h is bounded and has a log-Lipschitz MOC, we have $|h(x) - h(x+y)| \le \mu(|y|)$, where

$$\mu(r) = \begin{cases} -Mr\log r, & \text{if } |r| \le e^{-1}, \\ Me^{-1}, & \text{if } |r| > e^{-1}, \end{cases}$$

where $M = \left\|h\right\|_{LL}$. Thus, when $r \le e^{-1}$,

$$S_\mu(r) = -M\int_0^r \log s\,ds = M(r - r\log r).$$

Noting that $S_\mu(e^{-1}) = Me^{-1}$, when $r > e^{-1}$ we have

$$S_\mu(r) = S_\mu(e^{-1}) + \int_{e^{-1}}^r \frac{Me^{-1}}{s}\,ds = Me^{-1} + Me^{-1}(\log r - \log e^{-1}).$$

Further, when $r > e^{-1}$,

$$r\int_r^\infty \frac{\mu(s)}{s^2}\,ds = r\int_r^\infty \frac{Me^{-1}ds}{s^2} = Me^{-1}\frac{r}{r} = Me^{-1},$$

and when $r < e^{-1}$,

$$\begin{aligned} r \int_r^\infty \frac{\mu(s)}{s^2}\, ds &= -r \int_r^{e^{-1}} \frac{M \log s}{s}\, ds + r \int_{e^{-1}}^\infty \frac{M ds}{s^2} \\ &= -Mr \frac{1}{2} \left[(\log s)^2 \right]_r^{e^{-1}} + Mre^{-1} = \frac{M}{2} r \left[1 + (\log r)^2 \right] + Mre^{-1}. \end{aligned}$$

Applying Lemma 8.1, then, for $r > e^{-1}$,

$$\nu(r) = CM \left(\log r + 1 \right) \tag{8.2}$$

while for $r \le e^{-1}$,

$$\nu(r) = CMr \left[-\log r + (\log r)^2 \right],$$

which gives the MOC for q for small argument. □

Remark 8.3. As we can see from the proof of Lemma 8.2, the logarithmic bound on the growth of q at infinity comes from the L^∞-norm of h plus $S_\mu(e^{-1})$. Thus, such a logarithmic bound would hold for *any* h in $L^\infty(\mathbb{R}^2)$ as long as it also has *some* Dini MOC. Note, however, that $h \in L^\infty$, which would imply $q \in BMO$, is not by itself sufficient to obtain such a bound.

9. Afterword

We have characterized the behavior at infinity of 2D bounded solutions to the Euler equations in the full plane, including properties of the velocity and pressure, and have proved their existence and uniqueness. In the subsections that follow, we make three further observations: The first concerns a vorticity formulation of weak solutions; the second concerns the relation between our results and those of Taniuchi in [18] and Taniuchi, Tashiro, and Yoneda in [19]; the third concerns an extension of these results to the exterior of a single obstacle.

9.1. **Vorticity formulation of weak Solutions.** The definition of a weak solution to the 2D Euler equations for initial velocity in S given in [1] required that the solutions satisfy the Serfati identity, (2.2) (with $U_\infty \equiv 0$). This requirement was to insure uniqueness of solutions.

The Serfati identity encodes information both about the membership of the velocity field in S and the PDE (the Euler equations) that the velocity field satisfies. The renormalized Biot-Savart law of (1.4) only encodes the membership of the velocity field in a subspace of S for which the renormalized Biot-Savart law holds without taking a subsequence. It follows from Theorems 2.9 and 2.10 that we can use the renormalized Biot-Savart law—specifying the value of U_∞—instead of the Serfati identity as our selection criterion to insure uniqueness. This is more satisfying, as it reduces redundancy in the definition of a weak solution, and gives us the vorticity formulation of a weak solution in Definition 9.1, suitable for insuring both existence and uniqueness. Moreover, this definition is quite close to the usual vorticity formulation of solutions to the 2D Euler equations.

Definition 9.1 (Vorticity formulation of a weak solution in $\mathbb{R}^2$)**.** Fix $T > 0$ and $U_\infty \in C([0,T])$ with $U_\infty(0) = 0$. Let a be any radial cutoff function as in Definition 2.6. Let $u^0 \in S(\mathbb{R}^2)$ with vorticity $\omega^0 = \omega(u^0)$. We say that $\omega \in L^\infty([0,T] \times \mathbb{R}^2)$ is a *bounded solution* to the Euler equations without forcing having initial velocity u^0 and weak velocity at infinity U_∞ relative to u_0 if $u(0) = u^0$ and the following hold:

(1) $\partial_t \omega + K[\omega] \cdot \nabla \omega = 0$ as distributions on $(0,T) \times \mathbb{R}^2$, where $\omega = \omega(u)$;
(2) the velocity is recovered from the vorticity via
$$K[\omega](t) = u^0 + U_\infty(t) + \lim_{R \to \infty} (a_R K) * (\omega(t) - \omega^0), \tag{9.1}$$
and $K[\omega] \in C([0,T] \times \mathbb{R}^2)$;

A few comments on this definition:

(a) By Theorem 2.9, Definition 9.1 does not depend upon the particular choice of the radial cutoff function a.
(b) Even for a vorticity formulation, we must specify not just the bounded initial vorticity but the initial velocity, insisting that it too be bounded. This is because there are vorticity fields, ω^0, having no corresponding bounded velocity, u^0, a simple example being $\omega^0 \equiv 1$. Moreover, even if a u^0 exists it is unique only up to an additive constant.
(c) From $K[\omega] \in C([0,T] \times \mathbb{R}^2)$, the existence and uniqueness of a classical flow map follows as in Remark 2.4.
(d) The assumption that the velocity, $K[\omega]$, lie in $C([0,T] \times \mathbb{R}^2)$ seems to be necessary, as it does not follow from (9.1).

9.2. **Relation to work of Taniuchi, Tashiro, and Yoneda.** To construct his solutions to the Euler equations in [18], Taniuchi uses a sequence of approximating smooth solutions coming from [17]. In particular, he uses (2.5) (for $U_\infty \equiv 0$) to obtain the formula,
$$u(t_2) = u(t_1) - \int_{t_1}^{t_2} \mathcal{P}(u \cdot \nabla u)(t)\, dt,$$
where $\mathcal{P}$ is *formally* the Leray projector, defined in terms of Riesz transforms. This formula plays somewhat the same function that (2.2) plays in [1], and is central in Taniuichi's proof of existence of bounded (in fact, slightly unbounded) solutions.

Uniqueness of bounded solutions in the full plane is proved in [19]. Interestingly, the transport of the vorticity by the flow map is not used in [19] to prove uniqueness. They use the techniques of paradifferential calculus, along the lines of that of Vishik in [21], to obtain continuity with respect to initial data and hence uniqueness.

9.3. **Relation to the work of Jun Kato.** In [13], Jun Kato studies solutions to the Navier-Stokes equations in all of $\mathbb{R}^n$, $n \geq 2$ when the initial velocity is bounded. We restrict our comments here to the $n = 2$ case, where existence of solutions globally in time holds with
$$u \in L^\infty([0,T] \times \mathbb{R}^2) \text{ and } p = R_i R_j u^i u^j, \tag{9.2}$$
where $R_j = (-\Delta)^{\frac{1}{2}} \partial_j$ is a Riesz transform ([4, 15, 5, 12, 10]). Uniqueness was known to hold under the condition that (9.2) holds. The uniqueness condition was weakened somewhat in [11], then in [13] it was weakened quite a bit further to
$$u \in L^\infty([0,T] \times \mathbb{R}^2) \text{ and } p \in L^1_{loc}([0,T); BMO), \tag{9.3}$$

thereby dropping the requirement that the pressure satisfy any particular functional relation.

Kato employs in [13] a sequence of approximate Riesz operators, R^ε, converging to the Riesz transform R of Section 7 as $\varepsilon \to 0^+$, by cutting off the Green's function for the Laplacian. This same approach could have been taken here, since Lemma 8.1, which as at the heart of the proof of (2.8), holds uniformly when using R^ε in place of R. Instead of approximating the Riesz transform used to obtain the pressure, we, in Section 7, approximated the pressure itself. This has the virtue that it can, with substantial additional technical difficulties, be adapted to the exterior of a single obstacle. (We make a few comments on this in Section 9.4.)

A question that remains open is whether the condition in (2.6) can be dropped as long as (2.7) holds: this is what is done in [13] for the Navier-Stokes equations. What makes this difficult to prove for the Euler equations is that the Leray projector is not bounded in L^∞. For the Navier-Stokes equations, Kato gets around this by taking advantage of properties of the heat kernel. The key estimate, in Lemma 1 of [13], however, blows up like $(\nu t)^{-1/2}$, which prevents the estimate from being adapted for use with the Euler equations.

Finally, we note that the characterization at infinity in (1.3) can be extended to solutions to the Navier-Stokes equations with bounded initial velocity *and* vorticity. This is because the analog of the Serfati identity, (2.2), for the cutoff function, a_R, includes only the one additional term,

$$\nu \int_0^t \Delta_y \left((1 - a_R) K^j \right) * \omega(s) \, ds,$$

which vanishes as $R \to \infty$. This allows the argument in the proof of Proposition 5.1 to be made without change.

9.4. **Exterior to a single obstacle.** It is possible to obtain similar results for the exterior, Ω, to a single, simply connected obstacle having a $C^{2,\alpha}$ boundary, $\alpha \in (0,1)$. We give here a brief account of those results and comment on how they are obtained.

The main result, in analog with (1.3), is that

$$\begin{aligned} u(t,x) - u^0(x) &= U(t,x) + \lim_{R\to\infty} \int_\Omega a_R(x-y) J_\Omega(x,y) \omega(y) \, dy, \\ \nabla p(t,x) &= -\partial_t U(t,x) + O(1), \\ p(t,x) &= -\partial_t \zeta(t,x) + O(\log|x|). \end{aligned} \tag{9.4}$$

Here, J_Ω is the hydrodynamic Biot-Savart kernel (see [1]) and U is a bounded harmonic vector field (that is, divergence-free, curl-free, and tangential to the boundary), which is defined uniquely by its value, U_∞, at infinity and its circulation, γ, about the boundary. The function, γ, is the difference in the circulation of u^0 from that of $u(t)$. The vector field, ζ, and so the pressure, are multi-valued (unless $\gamma \equiv 0$) with $\nabla \zeta = U$. For physically meaningful solutions, we would require that $\gamma \equiv 0$, so that the pressure is single-valued and the circulation is unchanging.

The presence of an obstacle prevents us from transforming the vector field U (or even its value, U_∞, at infinity) away by making a change of reference frame, as we are able to do for the full plane. (Unless we wish to transform the problem to that of a moving obstacle.)

The proof of (9.4) parallels that given here for (1.3) but is substantially more technical and lengthy for the following reasons:

(1) Formulae involving convolutions in the full plane are replaced by integrals over Ω. Lemma 3.6, which allowed us to move derivatives back and forth in convolutions, must be replaced by integrating by parts, which introduces boundary terms that must be controlled. This complicates considerably the adaptation of the argument in Section 4 to an exterior domain.

(2) The presence of boundary terms also makes the analog of Lemma 5.4 for an exterior domain impossible to obtain. Instead, we need to strengthen the notion of a solution to require that

$$\int_\Omega (\varphi(t)\omega(t,\cdot) - \varphi(0)\omega^0(\cdot)) - \int_0^t \int_\Omega \partial_t \varphi \, \omega - \int_0^t \int_\Omega (\nabla\varphi \cdot u)\omega = 0$$

for all $\varphi \in C_C^\infty([0,T] \times \overline{\Omega})$, $t \in [0,T]$.

(3) The equivalent of Lemma 8.2 is much harder to obtain, and involves introducing a Neumann function (Green's function of the second kind) to solve for the pressure in terms of the velocity. This in turn requires the careful control of boundary integrals that do not appear for the full plane.

(4) The estimates on the hydrodynamic Biot-Savart kernel, J_Ω, corresponding to Proposition 3.1 are considerably harder to obtain than those for the Biot-Savart kernel, K, for the full plane. Fortunately, the needed estimates were obtained in [1].

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Department of Mathematics, University of California, Riverside, 900 University Ave., Riverside, CA 92521
E-mail address: `kelliher@math.ucr.edu`